

# ON THE COMBINATORIAL LOCAL LOG-CONCAVITY CONJECTURE AND A RESULT OF STANLEY

VALENTIN FÉRAY

The purpose of this note is to explain that the *combinatorial local log-concavity conjecture* introduced by Gross, Mansour, Tucker and Wang [GMTW15, Conjecture 3.1] in fact follows from a result of Stanley [Sta11, Corollary 3.3].

We use the following notation:

- $\lambda$  is a partition of  $n$  and  $\pi$  a fixed permutation of type  $\lambda$ .
- $Q_n$  is the set of  $n$ -cycles in the symmetric group  $S_n$ .
- $\rho(\sigma)$  denotes the cycle-type of  $\rho$ , while  $\kappa(\sigma)$  denotes the number of cycles of a permutation  $\sigma$ .
- $z_\lambda$  is the usual combinatorial factor, that is the size of the centralizer of  $\pi$  in  $S_n$ .

In his article [Sta11], Stanley considers the following polynomial

$$P_\lambda(q) = \sum_{\rho(w)=\lambda} q^{\kappa((1,\dots,n)w)},$$

while, in [GMTW15], the authors introduce

$$F_\lambda(q) = \sum_{\zeta \in Q_n} q^{\lfloor (\kappa(\zeta\pi) - 1)/2 \rfloor}.$$

The combinatorial local log-concavity conjecture states that for any partition  $\lambda$ , the polynomial  $F_\lambda$  is log-concave.

The polynomials  $P_\lambda$  and  $F_\lambda$  are in fact closely related, as shown by the following computation

$$\begin{aligned} P_\lambda(q) &= \sum_{\rho(w)=\lambda} q^{\kappa((1,\dots,n)w)} \\ &= \frac{1}{z_\lambda} \sum_{\sigma \in S_n} q^{\kappa((1,\dots,n)\sigma\pi\sigma^{-1})} \\ &= \frac{1}{z_\lambda} \sum_{\sigma \in S_n} q^{\kappa(\sigma^{-1}(1,\dots,n)\sigma\pi)} \\ &= \frac{n}{z_\lambda} \sum_{\zeta \in Q_n} q^{\kappa(\zeta\pi)}. \end{aligned}$$

Now we have two cases:

- either  $n + \kappa(\pi)$  is even, in which case  $\kappa(\zeta\pi)$  is odd for any  $n$ -cycle  $\zeta$  in  $Q_n$  and

$$P_\lambda(q) = \frac{n}{z_\lambda} q F_\lambda(q^2);$$

- or  $n + \kappa(\pi)$  is odd, in which case  $\kappa(\zeta\pi)$  is even for any  $n$ -cycle  $\zeta$  in  $Q_n$  and

$$P_\lambda(q) = \frac{n}{z_\lambda} q^2 F_\lambda(q^2).$$

From [Sta11, Corollary 3.3] (see also the discussion below this corollary), we know that  $P_\lambda(q)$  has only purely imaginary root, which implies in both cases that  $F_\lambda$  has (nonnegative) real roots. The log-concavity of its coefficients follows immediately.

#### REFERENCES

- [GMTW15] J.L. Gross, T. Mansour, T.W. Tucker and David G.L. Wang. Combinatorial Conjectures that imply local log-concavity of graph genus distributions. *European Journal of Combinatorics* **52** 207–222, 2016.
- [Sta11] Richard P. Stanley. Two enumerative results on cycles of permutations. *European Journal of Combinatorics*, **32** (6), 937–943, 2011.

INSTITUT FÜR MATHEMATIK, UNIVERSITÄT ZÜRICH, WINTERTHURERSTRASSE 190, 8057 ZÜRICH, SWITZERLAND

*E-mail address:* valentin.feray@math.uzh.ch